MULTISCALE PROCESS, MULTISCALE DECOMPOSITION, AND MULTISCALE DYNAMICS*

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CONTENTS

This is an unfinished draft. Citations should go to the published papers, say,

- X.S. Liang, 2016: Canonical transfer and multiscale energetics for primitive and quasigeostrophic atmospheres. J. Atmos. Sci., 73, 4439-4468 (*for canonical transfer, multiscale energetics, orthogonality, etc.*);
- Y. Liu, X.S. Liang, and R.H. Weisberg, 2007: Rectification of the bias in the wavelet power spectrum. J. Atmos. Ocean. Tech., 24, 2093-2102 (*for how to express the multiscale energy in wavelet spectra).*
- X.S. Liang and D.G.M. Anderson, 2007: Multiscale window transform. SIAM J. Multiscale Model. Simul., 6, 437-467 (*for multiscale energy expression, multiscale window transform and issues about filters*);
- X.S. Liang and A.R. Robinson, 2007: Localized multiscale energy and vorticity analysis: II. Finite-amplitude instability theory and validation. Dyn. Atmos. Oceans, 44, 51-76 (*for a connection of canonical transfer to baroclinic and barotropic instabilities in GFD*);

PART I. MULTISCALE ANALYSIS AND MULTI-SCALE ENERGY

1. Multiscale phenomena. Multiscale processes are ubiquitous in oceans and atmospheres. Related are many important yet notoriously difficult problems such as hydrodynamic instability, turbulence production, relaminarization, global climate change, storm growth, eddy shedding, to name but a few (Fig. 1.1). An in-depth understanding of multiscale processes is therefore crucial to the progress of geophysical fluid dynamics.

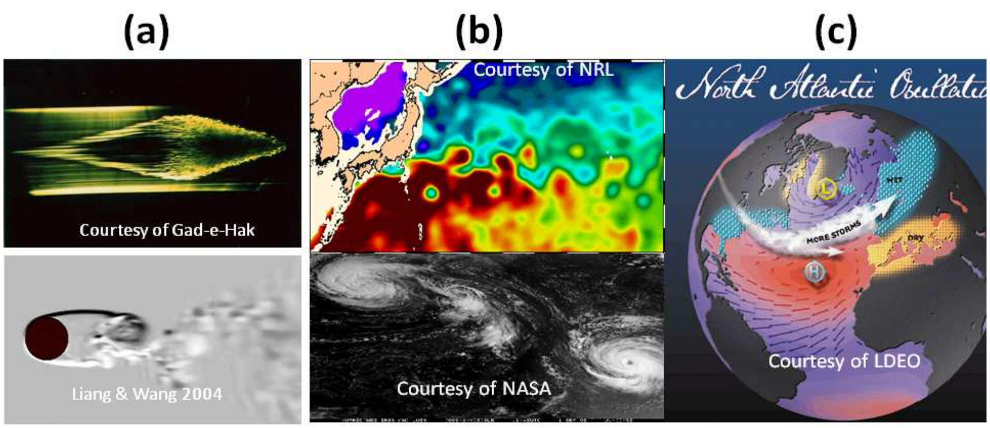

Fig. 1.1. *Examples of multiscale phenomena in fluid flows: (a) turbulence; (b) atmospheric cyclogenesis and ocean eddy shedding; (c) climate variability.*

We know energy is one of the most fundamental notions in physics. Central to the above problems is hence the transfer of energy between the processes as identified on different scales; see Fig. 1.3 for a schematic. In order to study the cross-scale energy transfer, we must first know how multiscale energy is represented, and then investigate what governs the evolution of the represented energy. These two form the themes of this lecture.

2. Concept of multiscale decomposition.

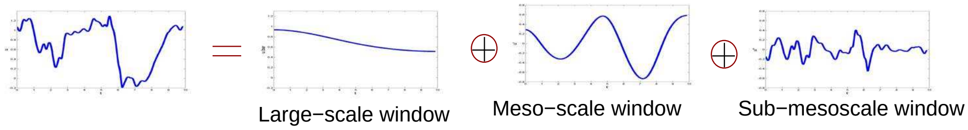

Fig. 1.2. *A superposition of three sub-processes on distinctly different scales results in a process at a point within the turbulence wake as shown in Fig. 1.1.*

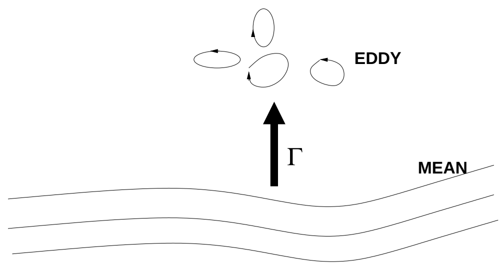

Fig. 1.3. *A schematic of the mean-eddy interaction, which is characterized by the energy transfer Γ between the mean and eddy processes.*

2.1. Fourier series expansion. We know from elementary calculus that, a function[1] $u(t)$,$t \in [0, 1]$, it can be expanded using the trigonometric series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2n\pi t) + b_n \sin(2n\pi t)], \tag{2.1}$$

where

$$a_0 = 2 \int_0^1 u(t) dt$$
$$a_n = 2 \int_0^1 u(t) \cos(2n\pi t) dt, \qquad n = 1, 2, ...$$
$$b_n = 2 \int_0^1 u(t) \sin(2n\pi t) dt, \qquad n = 1, 2, ...$$

Without loss of generality, here the function under consideration is assumed to be on $[0, 1]$; if not, we may always make it so by a coordinate transformation.

This expansion with trigonometric series, or Fourier series expansion, actually makes a multiscale decomposition most familiar to us. In doing so, $u(t)$ here is decomposed as a sum of infinite many terms, each with a distinct scale–here period $T_n = 1/n$, or angular frequency $\omega_n = 2n\pi$, for $n = 0, 1, 2, ...$ Note t may be replaced by spatial coordinate; correspondingly the scale is changed to spatial scale (wavelength).

2.2. Inner product space. We need to touch a little bit about Hilbert space or complete inner product space; we particularly need to consider the space $L_2[0, 1]$, which, loosely speaking, is a collection of square integrable functions on $[0, 1]$. It is equipped with an inner product $\langle \cdot, \cdot \rangle$ defined as, for any $u, v \in L_2[0, 1]$,

$$\langle u, v \rangle = \int_0^1 K(t) u(t) v(t) dt, \tag{2.2}$$

[1]For broad readership, here we avoid stating the conditions that u must satisfy, at a sacrifice of mathematical rigor.

with $K(t)$ being some given positive function of t or constant. The inner product allows a Hilbert space to have a geometric structure: $\langle u, v\rangle = 0 \Leftrightarrow u \perp v$. Besides, $\langle u, u\rangle = \|u\|^2$, i.e., the square of the norm $\|u\|$. We will soon see, this is actually the energy.

For a sequence $\{e_n\} \subset L_2[0,1]$, if any function $u \in L_2[0,1]$ can be expanded as a linear combination of $\{e_n\}$, then the sequence is called a basis of $L_2[0,1]$. Moreover, if $\{e_n\}$ is *orthonormal*, i.e.,

$$\begin{cases} \langle e_i, e_j\rangle = 0, & i \neq j, \\ \langle e_i, e_j\rangle = \|e_i\|^2 = 1. & i = j, \end{cases}$$

then from

$$u(t) = \sum_i \alpha_i e_i(t), \tag{2.3}$$

one has

$$\alpha_i = \langle u, e_i\rangle. \tag{2.4}$$

Eq. (2.4) is called a *transform* of u with basis $\{e_i\}$, and α_i the transform coefficients or *Fourier coefficients*. The expansion (2.3) is also called a *reconstruction* of u; particularly, any partial sum of the right hand side makes a *partial reconstruction* or, if no confusion arises, simply *reconstruction*, of u. Notice that reconstruction is a concept in physical space; here it is therefore a function of t. In contrast, transform coefficient is a concept in phase space; it is NOT a function of t.

To see this more clearly, the trigonometric expansion Eq. (2.1) can be re-written as

$$u(t) = \tilde{a}_0 + \sum_{n=1}^{\infty}[\tilde{a}_n(\sqrt{2}\cos\omega_n t) + \tilde{b}_n(\sqrt{2}\sin\omega_n t)] \tag{2.5}$$

where

$$\tilde{a}_0 = \int_0^1 u(t)\cdot 1 dt, \tag{2.6}$$

$$\tilde{a}_n = \int_0^1 u(t)\cdot(\sqrt{2}\cos\omega_n t)dt, \tag{2.7}$$

$$\tilde{b}_n = \int_0^1 u(t)\cdot(\sqrt{2}\sin\omega_n t)dt. \tag{2.8}$$

That is to say, $u(t)$ is actually expanded with the basis

$$\{1, e_{a,n}, e_{b,n}\} = \{1, \sqrt{2}\cos 2n\pi t, \sqrt{2}\sin 2n\pi t\},$$

which can be combined into

$$\{e_n\}_{n=0,1,2,...} = \{1, \sqrt{2}\cos 2\pi t, \sqrt{2}\sin 2\pi t, \sqrt{2}\cos 4\pi t, \sqrt{2}\sin 4\pi t, ...\}.$$

(In fact, this is simply $\{e^{i2n\pi t}\}$; we write so to avoid introducing complex functions which may complicate the introduction.) It is easy to prove that it is orthonormal with respect to the inner product defined in (2.2) with $K = 1$, and the coefficients $\tilde{a}_n = \langle u, e_{2n}\rangle,\ \tilde{b}_n = \langle u, e_{2n+1}\rangle,$ for $n = 0, 1, 2, ...$

If $\{e_n(t)\}$ is ordered by scale, just as that in the trigonometric series, the decomposition is a multiscale decomposition.

2.3. Decomposition by subspace. In real applications, more often than not we do not need to differentiate each scale; rather, it suffices to separate a process into sub-processes on exclusive ranges of scales. In formal language, we may decompose a space $V \in L_2[0,1]$ into a direct sum of several orthogonal subspaces, each on an exclusive range of scales. Such a subspace is termed by Liang and Anderson (2007) a *scale window*. That it to say, if we decompose $V = V_0 \oplus V_1 \oplus V_2$, with V_0, V_1, and V_2 representing, respectively, large-scale, meso-scale, and sub-mesoscale windows, then any function $u \in V$ can be decomposed as

$$u(t) = u^{\sim 0}(t) + u^{\sim 1}(t) + u^{\sim 2}(t) \tag{2.9}$$

with $u^{\sim\varpi} \in V_\varpi$, and $\varpi = 0, 1, 2$. Here we use $\sim\varpi$ in the superscripts to indicate that they are (partial) reconstructions on scale window ϖ, rather than on a particular scale. Eq. (2.9) is called a multiscale window decomposition.

For a function with a reconstruction as Eq. (2.3), one may combine certain terms to make a partial reconstruction on the scale window corresponding to those terms. But generally we may not have a reconstruction like (2.3). For example, application of some filter to u may result in a decomposition with a larger scale part and a smaller scale part:

$$u(t) = u_L(t) + u_H(t),$$

but we have no idea how each part is reconstructed as (2.3) does. In fact, (2.9) is similar to a decomposition through filtering, with $u^{\sim 0}, u^{\sim 1}, u^{\sim 2}$ similar to, respectively, the low-pass, band-pass, and high-pass parts. But they are also very different in that each $u^{\sim\varpi}$ corresponds to a transform coefficient which allows for energy representation, while traditional filters do not have this feature; we will see this soon in the following sections. Similarly, the Reynolds decomposition

$$u(t) = \bar{u} + u'(t)$$

is just a very special multiscale window decomposition.

3. Multiscale energy.

3.1. Multiscale energy is a concept in phase space — A common practice with filters is conceptually wrong. Multiscale energy is recalled every time when a power spectrum is examined (cf. Fig. 3.1c). In this sense, it should be most familiar to us. But, unfortunately, even about the notion of multiscale energy there exists misconception.

Suppose now we have a series

$$u(t) = a_0 \cos\omega_0 t + b_0 \sin\omega_0 t + a_1 \cos\omega_1 t + b_1 \sin\omega_1 t \tag{3.1}$$

with $\omega_0 \ll \omega_1$. We know, in the power spectrum, at these two scales $2\pi/\omega_0$ and $2\pi/\omega_1$ the energies are, respectively,

$$E_0 = a_0^2 + b_0^2 \tag{3.2}$$
$$E_1 = a_1^2 + b_1^2 \tag{3.3}$$
$$\tag{3.4}$$

(up to some constant factor such as $\frac{1}{2}$). So multiscale energy is a concept in phase space; it is function of the Fourier coefficients a_n and b_n which here are constants.

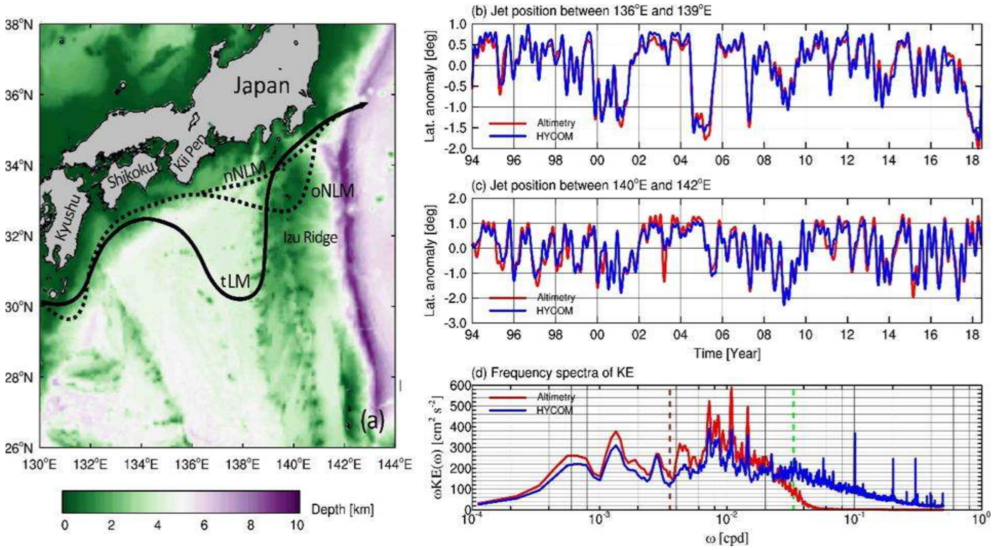

FIG. 3.1. *Time series and spectrum of the Kuroshio large meander: the time series of (a) the jet position between 130° E and 139° E, and (b) that between 140° E and 142° E; (c) the power spectra (averaged over the large meander region (30-34° N, 135-150° E) estimated from the altimetry data (red) and HYCOM (blue) (from Yang and Liang 2019).*

In the above example, the process has two distinctly different two scales. When a filter is used for multiscale decomposition, u can be separated into a low-pass part and a high-pass part:

$$\begin{aligned} u(t) &= u_L(t) + u_H(t), \\ u_L(t) &= a_0 \cos \omega_0 t + b_0 \sin \omega_0 t, \\ u_H(t) &= a_1 \cos \omega_1 t + b_1 \sin \omega_1 t. \end{aligned}$$

During the past 2-3 decades, it has been a common practice in this community, a can be found in vast publications in the literature, to simply take $[u_L(t)]^2$ and $[u_H(t)]^2$ as the large-scale energy and small-scale energy!

As easily seen, the common practice does not make sense at all. $[u_L(t)]^2$ and $[u_H(t)]^2$ are by no means equal to E_0 and E_1 in (3.2) and (3.3), respectively. Most of all, E_0 and E_1 are phase space variables (independent of t here), while $[u_L(t)]^2$ and $[u_H(t)]^2$ are in physical space (functions of t). So this common practice, which has been widely adopted due to its simplicity, is conceptually wrong.

3.2. Parseval's equality and orthonormality — Foundation of multiscale energy analysis. THEOREM 3.1.
Suppose V is a space in $L_2[0,1]$, and $\{e_n\}$ is an orthonormal basis of it. Let u, v be two functions in V. Then

$$\langle u, v \rangle = \sum_n \alpha_n \beta_n, \tag{3.5}$$

where $\alpha_n = \langle u, e_n \rangle$, $\beta_n = \langle v, e_n \rangle$ are the Fourier coefficients. If $u = v$, the above becomes

$$\sum_n \alpha_n^2 = \langle u, u \rangle = \|u\|^2. \tag{3.6}$$

Eqs. (3.5) and (3.6) are called **Parseval's equality** (or Parseval's relation, Parseval's identity). Sometimes Eq. (3.5) is also called the generalized Parseval equality. When the kernel $K = 1$, the right hand side of (3.6) is simply

$$\|u\|^2 = \int_0^1 u^2(t)dt = \overline{u^2(t)}, \tag{3.7}$$

i.e., the energy (up to some constant factor). So **Parseval's equality states that the total energy in physical space is equal to the sum of the energies on all scales**. This lays the foundation for multiscale energetics analysis and, in particular, for power spectral analysis.

So the square of the Fourier or transform coefficient, α_n^2, is the energy on the scale indexed by n. Moreover, for orthonormal basis $\{e_n\}$,

$$\alpha_n^2 = \alpha_n^2 \langle e_n, e_n\rangle = \langle \alpha_n e_n, \alpha_n e_n\rangle = \|\alpha_n e_n\|^2. \tag{3.8}$$

Since $\alpha_n e_n(t)$ is the reconstruction of u on the scale indexed by n, this is to say, **multiscale energy is actually the norm squared of the multiscale reconstruction**.

From above we know the importance of orthonormality of the basis. If not, the Parseval equality would not hold, and hence the so-obtained "multiscale energy" does not make sense in physics (energy would not be conserved).[2] The trigonometric basis (sine and cosine) is orthonormal and hence the Fourier power spectral analysis makes sense (in this case α_n^2 is $\tilde{a}_n^2 + \tilde{b}_n^2$), so does the widely used empirical orthogonal function (EOF) analysis.

The requirement of orthonormality allows us to interpret the Parseval equality (3.6) from a geometric aspect. Note $\alpha_n = \langle u, e_n\rangle$ is the projection of u onto the coordinate e_n. So $|\langle u, e_n\rangle|^2$ is the square of the side in that direction. So the Parseval equality actually states that the square of the hypotenuse, i.e., $\|u\|^2$, is equal to the sum of squares of the other sides. As is well known, this is the **Gougu Theorem** (Pythagorean theorem)!

3.3. Eddy energy with Reynolds decomposition—It is wrong if no averaging is taken. For a Reynolds decomposition $u = \bar{u} + u'$, in fluid dynamics we know that the corresponding mean energy and perturbation or eddy energy are

$$E_{mean} = \bar{u}^2, \tag{3.9a}$$

$$E_{eddy} = \overline{u'^2}, \tag{3.9b}$$

respectively. Notice that *there is a mean in the eddy energy expression.* With the mean there is no way to distinguish the eddy process at one location from that at another location on $[0, 1]$. In order to retain the local information, in the past decades there have appeared in the literature a lot of studies which simply remove the mean from the eddy energy expression. This is, again, conceptually wrong. (In fact, how to retain the local information is a much profound problem. It is by no means this trivial; see next section.)

To see why, first notice that (3.9) is consistent with the multiscale energy w.r.t. trigonometric basis. For

$$u(t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n e_n(t),$$

[2] For example, "spectral analyses" with non-orthonormal wavelet bases are problematic, which, however, have been widely used.

where we have symbolically written $\cos\omega_n t$ and $\sin\omega_n t$ into one $\{e_n\}$. Reynolds decomposition (originally in sample space; in practice usually performed in time):

$$u(t) = \bar{u} + u'(t)$$

means

$$\bar{u} = \int_0^1 u(t)dt = \alpha_0 = \text{const}, \tag{3.10}$$

$$u'(t) = \sum_{n=1}^{\infty}[\alpha_n e_n(t)]. \tag{3.11}$$

Obviously, $\bar{u}^2 = \alpha_0^2$, i.e., mean energy or energy on scale $\omega_0 = 0$. By Parseval's equality,

$$\sum_{n=1}^{+\infty} \alpha_n^2 = \int_0^1 u'^2(t)dt = \overline{u'^2(t)}. \tag{3.12}$$

The left hand side is the total energy for all scales ω_n, $n = 1, 2, ...$, excluding the mean, i.e., the eddy energy. So the averaging on the right hand side is essential! One cannot have it removed to gain "local eddy energy"!

3.4. Multiscale energy with respect to filtering—Orthogonality is pivotal. For a filtered field with varying background: $u = u_L(t) + u_H(t)$, what are the energies corresponding to the low-pass and high-pass fields? Previously it has been a common practice to simply take them as $u_L^2(t)$ and $u_H^2(t)$. As we explained in the beginning in this section, this is conceptually wrong. So how can we find them?

By the foregoing argument, the energy for the low-pass field is $\|u_L\|^2$, and that for the high-pass field is $\|u_H\|^2$. In order for them to sum to the total energy $\|u\|^2$, i.e., in order to have

$$\|u\|^2 = \|u_L\|^2 + \|u_H\|^2$$

there decomposition must be orthogonal (Gougu theorem/Pythagorean theorem)! So **NOT just any filter works for a physically faithful multiscale decomposition**—orthogonality is pivotal!

Note here we have only considered the total energy for each scale/scale window. To find the energy on that scale/scale window localized at some location in physical space, it is still a big challenge. This will be introduced in the following section.

4. Time-frequency/space-wavenumber decomposition.

4.1. The difficulty with the traditional multiscale decomposition techniques. In the preceding section we have introduced the multiscale energy representation with respect to the traditional techniques such as Fourier analysis and Reynolds decomposition, which are *global* in that the resulting multiscale energy does not have local information retained. This global nature may encounter problems in dealing with realistic atmosphere-ocean processes. For example, Fig. 4.1a is the time series of the temperature at 10hPa over the North Pole. From it we see that, during some short periods of the winters of 2011/12, 2012/13, 2014/15, the temperature can be as high as its summer value. This phenomenon is called *sudden stratospheric warming*. If we

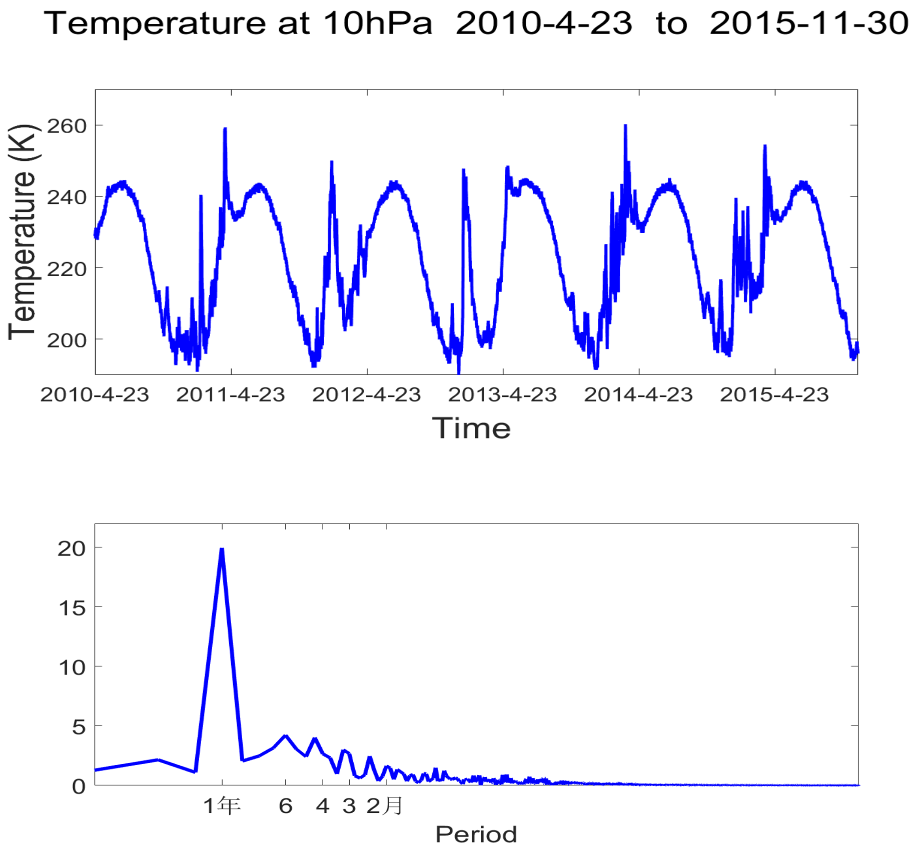

Fig. 4.1. *The time series and Fourier power spectrum of the 10-hPa temperature over the North Pole. In the spectrum the sudden warming event is not seen.*

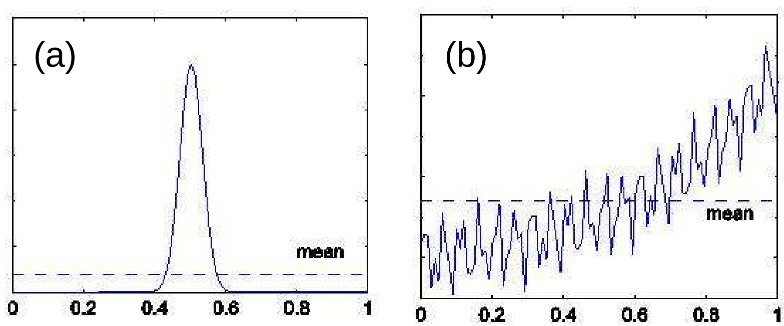

Fig. 4.2. *Reynolds decomposition applied to (a) an energy burst process, and (b) a process with nonstationary background.*

check the Fourier power spectrum, as shown in Fig. 4.1b, however, this conspicuous abnormal warming event is absent.

The Reynolds decomposition does not retain local information retained, either. For the energy burst process as shown in Fig 4.2a, by subtracting the mean from the original series, we see perturbations everywhere on [0,1], while in reality nothing happens outside the burst region. The resulting eddy energy is hence a constant over [0,1]; one cannot distinguish the motionless region from the energy burst region. Another example (Fig 4.2b) regards the perturbation from a nonstationary background. By taking deviation from the mean, as indicated in the figure, the so-obtained perturbation is unrealistically large, much larger than that one can identify via visual inspection.

4.2. Multiscale energy with orthonormal wavelets—Wavelet spectra are NOT "biased" at all. The above difficulty with the traditional analysis techniques has been identified as be a major challenge many decades agao. In 80's and 90's, there has been a surge of interest in developing localized analysis methods to address the problem, e.g., wavelet transform, Hilbert-Huang transform, etc. In the following we use wavelets to briefly illustrate how multiscale energy is represented with a localized transform. Details are referred to Liu et al. (2007) and Liang and Anderson (2007).

As before, consider a function $u(t)$ defined on $[0,1]$. Suppose we have a mother wavelet ψ and hence a wavelet basis

$$\psi_n^j(t) = 2^{j/2}\psi(2^j t - n), \qquad t \in (-\infty, \infty) \tag{4.1}$$

where the positive integer $j \geq 0$ is scale level (2^{-j} the scale), and integer n is a variable in sampling space, corresponding to the physical space location. $\{\psi_n^j(t)\}$ is different from the traditional basis in that it is embedded with both dilation (j) and translation (n); a multiscale analysis with respect to it hence can have local information retained, in contrast to that with respect to, say, trigonometric basis. Further suppose that $\{\psi_n^j\}$ is orthonormal with respect to j and n in $L_2(-\infty,\infty)$. Recall u is on $[0,1]$, which must be extended to the whole real line. There are many ways to fulfill this; see Liang and Anderson (2007) for examples which are satisfactory in practice. Here for illustration's sake we just consider one of them, periodization.

It has been established that, after periodization, $\{\phi_n^j(t)\}_{t\in\mathbb{R}}$ yields a periodized basis $\{\tilde{\phi}_n^j(t)\}_{t\in[0,1]}$, and now n takes values on a finite set $(0,1,...,2^j-1)$; detailed expression is referred to Liang and Anderson (2007, section 2.3). Furthermore, it has been proved that $\{\tilde{\phi}_n^j(t)\}$ is also orthonormal in $L_2[0,1]$ with respect to j and n.

So the reconstruction on a specific scale 2^{-j} (or at scale level j) is

$$u^j(t) = \sum_{n=0}^{2^j-1} \hat{u}_n^j \tilde{\psi}_n^j(t), \qquad j > 0,\ t \in [0,1], \tag{4.2}$$

where

$$\hat{u}_n^j = \left\langle u, \tilde{\psi}_n^j \right\rangle = \int_0^1 u(t)\tilde{\psi}_n^j(t)dt \tag{4.3}$$

is the transform coefficient. Here we only consider the cases $j > 0$. For $j = 0$ with a periodized basis, it is simply the mean $\bar{u}$, and the energy is $(\bar{u})^2$, as with the Reynolds decomposition.

By the orthonormality of $\{\tilde{\psi}_n^j\}$, the Parseval equality (3.6) implies,

$$\sum_{n=0}^{2^j-1} [\hat{u}_n^j]^2 = \int_0^1 [u^j(t)]^2 dt = \overline{[u^j(t)]^2}. \tag{4.4}$$

The right hand side (r.h.s.) is the mean energy of u at level j. (Again the mean is essential.) The l.h.s. is the sum of $N = 2^j$ parts, each representing the process on a small interval D_n centered around $t_n = 2^{-j}n$, with a length of

$$\Delta t = 1/N = 2^{-j}.$$

The r.h.s. can also be written as a sum of N parts,

$$\overline{[u^j(t)]^2} = \int_0^1 [u^j(t)]^2 dt \sum_{n=0}^{2^j-1} \int_{t_n}^{t_n+\Delta t} [u^j]^2 dt = \sum_{n=0}^{2^j-1} \overline{[u^j(t)]^2}^{D_n} \cdot \Delta t. \tag{4.5}$$

Note $\overline{[u^j(t)]^2}^{D_n}$ is the mean energy on scale 2^{-j} over D_n, i.e., the interval around $t_n = 2^{-j}n$. (Note the mean!) Denote it by E_n^j. Comparing (4.4) and (4.5), we have (Liu et al., 2007)

$$E_n^j = \frac{1}{\Delta t}[\hat{u}_n^j]^2 = 2^j \times [\hat{u}_n^j]^2. \tag{4.6}$$

Figure 4.3 is the wavelet power spectrum of the time series in Fig. 4.1a. In contrast to its Fourier counterpart (Fig. 4.1b), the sudden warming events are clearly seen.

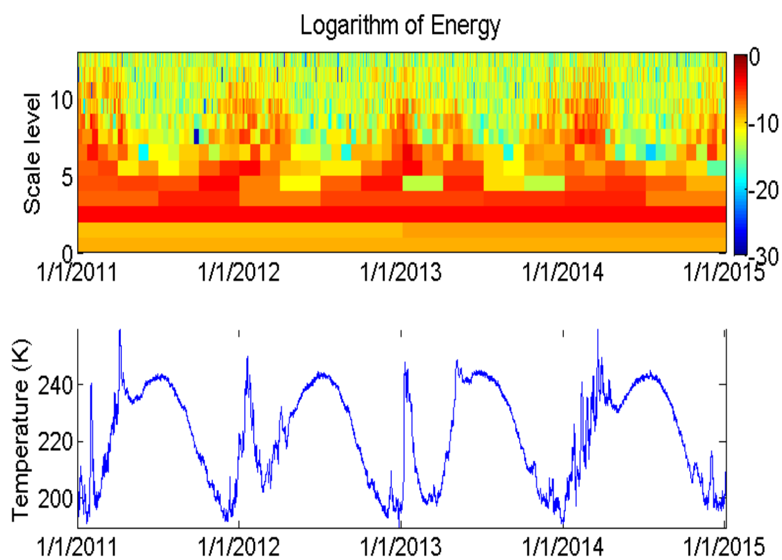

FIG. 4.3. *The wavelet power spectrum corresponding to Fig. 4.1 with respect to the orthonormal wavelet basis constructed in Liang and Anderson (2007). The sudden warming event makes one of most conspicuous features in the spectrum, in contrast to that in Fig. 4.1.*

Historically, it has long believed that wavelet spectra are "biased" in that the resulting multiscale energy is not consistent with that from Fourier power spectra; a lot of studies have been conducted to address the "issue." This is because people usually take as the square of the transform coefficient as the energy on the corresponding scale, just as Torrence and Compo (1998) did in their wavelet spectral analysis package, a widely used software package. This is, unfortunately, NOT true. By (4.6), **the energy on a scale with an orthonormal wavelet basis is not the square**

of the transform coefficients, but the square of the transform coefficients divided by the corresponding scale!

Liu et al. (2007) showed that the wavelet power spectra based on Eq. (4.6) are consistent with their Fourier counterparts. The so-called "bias" issue is hence NOT an issue at all. What we should bear in mind is that the Parseval equality, from which we have derived (4.6), is the key; it lays the ground for multiscale energy analysis.

Notice that, just as before, the Parseval equality requires that the basis be orthonormal. So power spectral analysis with a non-orthonormal basis (just as that in Torrence Compo, 1998) generally does not make sense in physics.

4.3. Scale window and multiscale window transform. More often than not, oceanic and atmospheric processes tends to occur on a range of scales, or scale windows, rather than some individual scale. For example, the ocean eddies may have a life cycle from several days to half a year; the Madden-Julian Oscillation (MJO) has a broadband spectrum between 30 and60 days. So, as mentioned before, we often need to consider scale window decomposition.

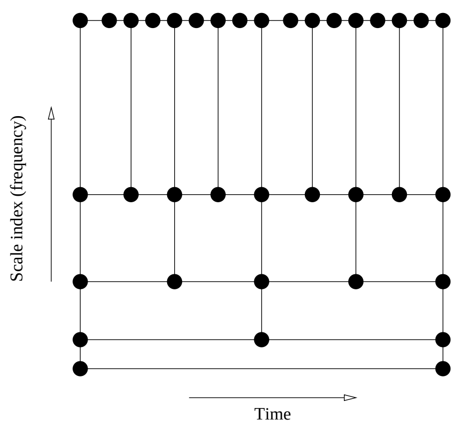

FIG. 4.4. *Schematic of the time-frequency representation of an orthonormal wavelet transform. The transform coefficients, and hence energies, are defined discretely at different locations for different scale levels (from Liang and Anderson 2007).*

The scale windows of some space in $L_2[0,1]$ are mutually orthogonal subspaces, each with an exclusive range of scales. Note orthogonality is pivotal here; otherwise we would not be able to talk about energy. One may tend to think that a straight way to construct a scale window is simply to make direct sum across the scale levels in a wavelet spectrum. But there is an issue here. As schematized in Fig. 4.4, the energies on different scales are defined discretely at different locations. It is impossible to study the energy at a particular location over a range of scales, as they cannot be summed together across the scales at that location. (One cannot rely on interpolation as the energies are discretely defined at the points.) So we have to seek other ways to the problem. The following is extracted from Liang (2016), §2.

"Consider a Hilbert space $V_{\ell,j} \subset L_2[0,1]$ generated by the basis $\{\phi_n^j(t)\}_{n=0,1,...,2^j\ell-1}$, where

$$\phi_n^j(t) = \sum_{q=-\infty}^{+\infty} 2^{j/2}\phi[2^j(t+\ell q) - n + 1/2], \qquad n = 0, 1, ..., 2^{j\ell-1}. \tag{4.7}$$

Here $\phi(t)$ is a scaling function constructed in Liang and Anderson (2007) such that $\{\phi(t-n+1/2)\}_n$ is orthonormal (Fig. 4.5). The parameter $\ell = 1$ or $\ell = 2$, corresponding respectively to the periodic and symmetric extension schemes. Shown in Fig. 4.6 is the basis for $\ell = 2$ and a selection of j, namely, the "scale level" (2^{-j} is the scale). For notational simplicity, throughout this study the dependence of ϕ_n^j on ℓ is suppressed (but retained in other notations).

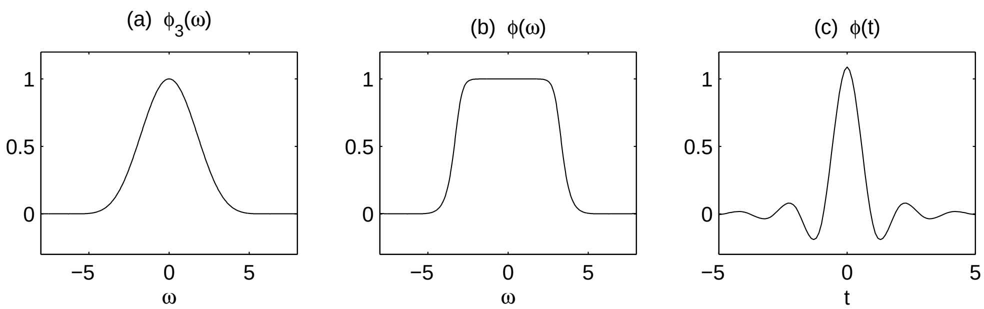

FIG. 4.5. *The orthonormal scaling function ϕ in frequency domain (middle) and time domain (right) constructed from the cubic spline (left) (Liang and Anderson, 2007).*

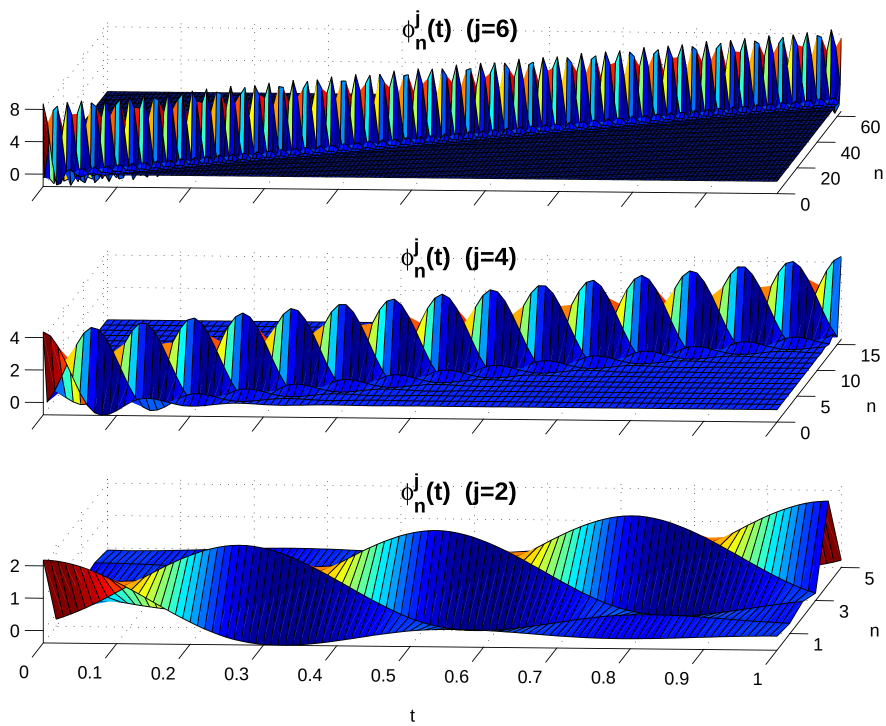

FIG. 4.6. *ϕ_n^j as function of t (physical space variable; here time) and n (sampling space variable) for a selection of scale level j (from Liang 2016).*

It has been justified in Liang and Anderson (2007) that there always exists a j_2

such that all the atmospheric/oceanic signals of concern lie in V_{ℓ,j_2}. Furthermore, it has been shown in there that

$$V_{\ell,j_0} \subset V_{\ell,j_1} \subset V_{\ell,j_2}, \qquad \text{for } j_0 < j_1 < j_2.$$

A decomposition thus can be made such that

$$V_{\ell,j_2} = V_{\ell,j_1} \oplus W_{\ell,j_1-j_2} = V_{\ell,j_0} \oplus W_{\ell,j_0-j_1} \oplus W_{\ell,j_1-j_2} \tag{4.8}$$

where W_{ℓ,j_1-j_2} is the orthogonal complement of V_{ℓ,j_1} in V_{ℓ,j_2}, and W_{ℓ,j_0-j_1} that of V_{ℓ,j_0} in V_{ℓ,j_1}. It has been shown by Liang and Anderson (2007) that V_{ℓ,j_0} contains functions of scales larger than 2^{-j_0} only, while lying in W_{ℓ,j_0-j_1} and W_{ℓ,j_1-j_2} are the functions with scale ranges between 2^{-j_0} to 2^{-j_1} and 2^{-j_1} to 2^{-j_2}, respectively. We call the so-formed subspaces of V_{ℓ,j_2} as scale windows. For easy reference, from larger scales (lower scale levels) to smaller scales (higher scale levels), they will be referred to as scale windows 0, 1, and 2, respectively. Depending on the problem of concern, they may also be assigned names in association to physical processes. For example, one may refer to them as large-scale, mid-scale, and small-scale windows, or, in the context of, say, MJO studies, mean window, intraseasonal window or MJO window, and synoptic window, or, in the context of oceanography, large-scale window, mesoscale window and sub-mesoscale window, etc. More scale windows can be likewise defined, but usually three are enough (in fact in many cases only two are needed).

Consider a function $u(t) \in V_{\ell,j_2}$. With (4.7), a transform

$$\widehat{u}_n^j = \int_0^\ell u(t)\phi_n^j(t)\ dt, \tag{4.9}$$

can be defined for a scale level j. Given window bounds $j_0 < j_1 < j_2$, u then can be reconstructed on the three scale windows as constructed above:

$$u^{\sim 0}(t) = \sum_{n=0}^{2^{j_0}\ell-1} \widehat{u}_n^{j_0}\phi_n^{j_0}(t), \tag{4.10}$$

$$u^{\sim 1}(t) = \sum_{n=0}^{2^{j_1}\ell-1} \widehat{u}_n^{j_1}\phi_n^{j_1}(t) - u^{\sim 0}(t), \tag{4.11}$$

$$u^{\sim 2}(t) = u(t) - u^{\sim 0}(t) - u^{\sim 1}(t), \tag{4.12}$$

with the notations $\sim$0, $\sim$1, and $\sim$2 signifying respectively the corresponding three scale windows. Since V_{ℓ,j_0}, W_{ℓ,j_0-j_1}, W_{ℓ,j_1-j_2} are all subspaces of V_{ℓ,j_2}, the functions $u^{\sim 0}$, $u^{\sim 1}$, $u^{\sim 2}$ can be transformed with respect to $\{\phi_n^{j_2}(t)\}_n$, the basis of V_{ℓ,j_2},

$$\widehat{u}_n^{\sim\varpi} = \int_0^\ell u^{\sim\varpi}(t)\ \phi_n^{j_2}(t)\ dt, \tag{4.13}$$

for windows $\varpi = 0, 1, 2$, and $n = 0, 1, ..., 2^{j_2\ell} - 1$. Note here the transform coefficients $\widehat{u}_n^{\sim\varpi}$ contains only the processes belonging to scale window ϖ. It has, though discretely, the finest resolution permissible in the sampling space on $[0, 1]$. We call (4.13) a *multiscale window transform*, or MWT for short. With this, (4.10), (4.11), and (4.12) can be written in a unified way:

$$u^{\sim\varpi}(t) = \sum_{n=0}^{2^{j_2}\ell-1} \widehat{u}_n^{\sim\varpi}\phi_n^{j_2}(t), \qquad \varpi = 0, 1, 2. \tag{4.14}$$

Eqs. (4.13) and (4.14) form the transform-reconstruction pair for MWT.

It can be proved that, for any $u \in L_2[0,1]$,

$$u^{\sim 0}(t) = \bar{u} = \text{const}\,.$$

So Reynolds decomposition is a very special MWT with two windows and $j_0 = 0$ (and a periodic extension is adopted)."

4.4. Multiscale energy with MWT. The following is from Liang (2016), section 2b.

"MWT has a Parseval relation-like property; in the periodical extension case ($\ell = 1$),

$$\sum_n \widehat{u}_n^{\sim\varpi} \widehat{v}_n^{\sim\varpi} = \overline{u^{\sim\varpi}(t)\ v^{\sim\varpi}(t)}, \tag{4.15}$$

for $u, v \in V_{1,j_2}$, and because of the mutual orthogonality between the scale windows,

$$\sum_\varpi \sum_n \widehat{u}_n^{\sim\varpi} \widehat{v}_n^{\sim\varpi} = \overline{u(t)v(t)}, \tag{4.16}$$

where the overline indicates averaging over time, and $\sum_n$ is a summation over the sampling set $\{0, 1, 2, ..., 2^{j_2} - 1\}$ (see Liang and Anderson (2007) for a proof). In the case of other extensions, $\sum_n$ is replaced by "marginalization", a naming convention after Huang et al. (1999), which also bears the physical meaning of summation over n. Eq. (4.16) states that, a product of two MWT coefficients followed by a marginalization is equal to the product of their corresponding reconstructions averaged over the duration. This property is usually referred to as property of marginalization.

The property of marginalization is important in that it allows for an efficient representation of multiscale energy in terms of the MWT transform coefficients. In (4.16), let $u = v$, the right hand side is then the energy of u (up to some constant factor) averaged over $[0, 1]$. It is equal to a summation of $3N = 3 \times 2^{j_2}$ (if 3 scale windows are considered) individual objects $(\widehat{u}_n^{\sim\varpi})^2$ centered at time $t_n = 2^{-j_2}n + 1/2$, with a characteristic influence interval $\Delta t = t_{n+1} - t_n = 2^{-j_2}$. The multiscale energy at time t_n then should be the mean over the interval: $\frac{(\widehat{u}_n^{\sim\varpi})^2}{\Delta t} = 2^{j_2}(\widehat{u}_n^{\sim\varpi})^2$. Notice the constant multiplier 2^{j_2}; it is needed for the obtained multiscale energy to make sense in physics. But for notational succinctness, it will be omitted in the following derivations.

Therefore, the energy of u on scale window ϖ at step n is

$$E_n^\varpi \propto (\widehat{u}_n^{\sim\varpi})^2\,. \tag{4.17}$$

Note the ϖ-window filtered signal is $u^{\sim\varpi}$; by the common practice one would take $(u^{\sim\varpi})^2$ as the energy on scale window ϖ. From above one sees that this is conceptually incorrect."

PART II. MULTISCALE ENERGETICS—EDDY GENERATION, MEAN-EDDY INTERACTION, AND TURBULENCE PRODUCTION

5. Issues with the classical empirical formalism. The mean flow-eddy interaction is a fundamental problem in dynamic meteorology and physical oceanography. Related to it are emergence of coherent structure, hydrodynamic stability, turbulence production, relaminarization, atmospheric cyclogenesis, hurricane generation, ocean eddy shedding, to name a few. Central to the problem is the transfer of energy between the mean and eddy processes as decomposed (cf. Fig. 1.3). Historically there have been two types of multiscale energetics, one in wavenumber space (e.g., Saltzman, 1957), another in ensemble space. The former allows for us to examine the evolution of wavenumber spectra as coherent structures emerge and decay. The disadvantage is that physical meaning is not easy to see in the complex mathematical expressions of the nonlinear interactions between the spatial scales. On the other hand, the statistical formalism with ensemble mean, which in practice is usually replaced by time mean, has clear physical interpretations for each of the resulting terms (Lorenz, 1955). In this lecture, we only consider this type of formalism, or Lorenz-type formalism.

5.1. Traditional cross-scale energy transfer with Reynolds decomposition. The classical Lorenz-type formalism of energy transfer can be best illustrated with the Reynolds decomposed equations for the advection of a scalar field $T = \bar{T} + T'$ in an incompressible flow $\mathbf{v}$, where the overbar stands for an ensemble mean, and the prime for the departure from the mean. In the absence of diffusion, T evolves as

$$\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{v}T) = 0, \tag{5.1}$$

whose decomposed equations are

$$\frac{\partial \bar{T}}{\partial t} + \nabla \cdot (\bar{\mathbf{v}}\bar{T} + \overline{\mathbf{v}'T'}) = 0, \tag{5.2a}$$

$$\frac{\partial T'}{\partial t} + \nabla \cdot (\mathbf{v}'\bar{T} + \bar{\mathbf{v}}T' + \mathbf{v}'T' - \overline{\mathbf{v}'T'}) = 0. \tag{5.2b}$$

Multiplying (5.2a) by $\bar{T}$, and (5.2b) by T', and taking the mean, one arrives at the evolutions of the mean energy and eddy energy (variance) (e.g., Pope 2003)

$$\frac{\partial \bar{T}^2/2}{\partial t} + \nabla \cdot (\bar{\mathbf{v}}\bar{T}^2/2) = -\bar{T}\nabla \cdot (\overline{\mathbf{v}'T'}) \tag{5.3a}$$

$$\frac{\partial \overline{T'^2/2}}{\partial t} + \nabla \cdot (\overline{\mathbf{v}T'^2/2}) = -\overline{\mathbf{v}'T'} \cdot \nabla\bar{T}. \tag{5.3b}$$

The terms in divergence form are generally understood as the transports of the mean and eddy energies, and those on the right hand side as the respective energy transfers. The latter are usually used to explain the mean-eddy interaction. Particularly, when T is a velocity component, the right hand side of (5.3b) has been interpreted as the rate of energy extracted by Reynolds stress, or "Reynolds stress extraction" for short, against the mean field to fuel the eddy growth; in the context of turbulence research, it is also referred to as the "rate of turbulence production".

An observation of the two "transfer terms" on the right hand sides of (5.3) is that they are not symmetric; in other words, they do not cancel out each other. In fact, they sum to $\nabla \cdot (\bar{T}\overline{\mathbf{v}'T'})$, which in general does not vanish. This is not what one expects, as physically a transfer process should be a mere redistribution of energy between the mean and eddy processes, without destroying or generating energy as a whole. These two quantities therefore are not real transfers, and cannot be used to measure the mean-eddy interaction.

5.2. Barotropic and baroclinic instabilities. The above problem arises from the empirical definition of multiscale flux, which is quite ambiguous. For example, can we say $\bar{\mathbf{v}}\bar{T}^2/2$ is the flux of mean energy and $\overline{\mathbf{v}T'^2/2}$ that of eddy energy? The answer is not that clear. As argued by Plumb (1983), there could be other kinds of fluxes which may result in totally different energy transfers. One way to overcome this difficulty is to average/integrate all the energetics over a closed domain. In doing this the divergences of fluxes that may be intertwined with the nonlinear term will be gone, leaving only the transfers.

This is actually how it is treated in Pedlosky (1979). Consider a perturbation flow $\mathbf{v}'$ and perturbation density ρ', and let $c = g^2/\rho_0^2 N^2$ (N is the buoyancy frequency). Then there will be an evolution equation like (5.3b) for eddy energy

$$E_{eddy} = \frac{1}{2}\overline{\mathbf{v}' \cdot \mathbf{v}'} + \frac{1}{2}c\overline{\rho'^2}. \tag{5.4}$$

Since the flux terms (transport) and transfers are difficult to be separated, a spatial averaging over the whole domain is performed. The resulting equation is

$$\frac{\partial \langle E_{eddy} \rangle}{\partial t} = \left\langle -c\overline{v'\rho'}\frac{\partial \bar{\rho}}{\partial y} - c\overline{w'\rho'}\frac{\partial \bar{\rho}}{\partial z} \right\rangle + \left\langle -\overline{u'v'}\frac{\partial \bar{u}}{\partial y} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} \right\rangle, \tag{5.5}$$

where the angle brackets indicate spatial averages over the whole domain. In Pedlosky (1979), the eddy energy growth due to that in the first angle bracket is called baroclinic instability, while that due the the second angle bracket is called barotropic instability. Note in the case of quasi-geostrophic flow, the terms with w' are all gone, and hence the first angle bracket is only related to $\frac{\partial \bar{\rho}}{\partial y}$, which by thermal wind relation is proportional to $\frac{\partial \bar{u}}{\partial z}$; similarly the second angle bracket is only related to $\frac{\partial \bar{u}}{\partial y}$. That is to say, for quasi-geostrophic flows, barotropic and baroclinic instabilities are, respectively, due to horizontal and vertical shears of the basic flow. But generally this does not need to be the case.

The concept of barotropic and baroclinic instabilities is very important in geophysical fluid dynamics. Eq. (5.5), however, is expressed in both temporal mean and spatial mean, and hence cannot be used to investigate those processes such as coherent structure emergence, cyclogenesis, eddy shedding, etc., which are in nature highly localized in space and intermittent in time. Note that, as we elaborated in section 3.3, the overbars (mean) in the equation are essential for multiscale energetics; it is conceptually wrong to remove them to achieve temporal localization (as in many studies in the literature). On the other hand, the angle brackets (spatial averaging) are also important; without them the energy transfers may not be faithful.

To allow barotropic instability and baroclinic instability to play their due roles in the studies of dynamic oceanography and meteorology, (5.5) must be localized. But the localization is by no means as trivial as to get rid of the temporal and spatial averages. The key issues here are:

- whether there is a localized multiscale decomposition technique to replace the Reynolds decomposition;

- whether the transport and transfer processes can be unambiguously separated.

The following sections are devoted to answering these issues.

6. Multiscale transport and canonical transfer.

6.1. Multiscale flux with MWT. Within the framework of MWT, multiscale flux can be rigorously derived, and hence the transport-transfer separation is made unique. This section is extracted from Liang (2016), section 3a.

"As we showed in §4.4, for a scalar field T, its energy (quadratic property) on window ϖ at step n is $\frac{1}{2}(\widehat{T}_n^{\sim\varpi})^2$ (up to some factor). In the MWT framework, energy can be decomposed as a sum of a bunch of atom-like elements:

$$\frac{1}{2}T^2 = \sum_{n_1,\varpi_1}\sum_{n_2,\varpi_2}\frac{1}{2}\left[\widehat{T}_{n_1}^{\sim\varpi_1}\phi_{n_1}^{j_2}(t)\right]\left[\widehat{T}_{n_2}^{\sim\varpi_2}\phi_{n_2}^{j_2}(t)\right]. \tag{6.1}$$

Look at the flux of the "atom" by a flow $\mathbf{v}(t)$ over $t\in[0,1]$ at step n within window ϖ. It is

$$\int_0^1 \mathbf{v}(t)\cdot\frac{1}{2}\left[\widehat{T}_{n_1}^{\sim\varpi_1}\phi_{n_1}^{j_2}(t)\right]\left[\widehat{T}_{n_2}^{\sim\varpi_2}\phi_{n_2}^{j_2}(t)\right]\cdot\delta(n-n_2)\cdot\delta(\varpi-\varpi_2)\ dt. \tag{6.2}$$

In the above delta functions, the arguments may equally be chosen as n_1 and ϖ_1. The flux of $\frac{1}{2}T^2$ by the flow $\mathbf{v}$ on ϖ at step n is then the sum of the atomic expressions over all the possible n_1, n_2, ϖ_1, and ϖ_2, i.e.,

$$\begin{aligned}\mathbf{Q}_n^{\varpi} &= \sum_{n_1,\varpi_1}\sum_{n_2,\varpi_2}\int_0^1\frac{1}{2}\mathbf{v}\cdot\left[\widehat{T}_{n_1}^{\sim\varpi_1}\phi_{n_1}^{j_2}(t)\right]\left[\widehat{T}_{n_2}^{\sim\varpi_2}\phi_{n_2}^{j_2}(t)\right]\cdot\delta(n-n_2)\delta(\varpi-\varpi_2)dt\\ &= \frac{1}{2}\int_0^1\mathbf{v}(t)T(t)\cdot\widehat{T}_n^{\sim\varpi}\phi_n^{j_2}(t)dt.\end{aligned} \tag{6.3}$$

But the function $\widehat{T}_n^{\sim\varpi}\phi_n^{j_2}(t)$ lies in window ϖ, and all windows are orthogonal, so this is something like a projection of $\mathbf{v}T$ onto window ϖ:

$$\begin{aligned}\mathbf{Q}_n^{\varpi} &= \frac{1}{2}\int_0^1\widehat{(\mathbf{v}T)}_n^{\sim\varpi}\cdot\widehat{T}_n^{\sim\varpi}\phi_n^{j_2}(t)dt\\ &= \frac{1}{2}\widehat{T}_n^{\sim\varpi}\widehat{(\mathbf{v}T)}_n^{\sim\varpi}.\end{aligned} \tag{6.4}$$

The above can be used for the derivation of multiscale potential energetics. For kinetic energy $K=\frac{1}{2}\mathbf{v}\cdot\mathbf{v}$, essentially one can derive in the same way. To avoid confusion, we consider the energy-like quantity of an arbitrary vector $\mathbf{G}$,

$$K=\frac{1}{2}\mathbf{G}\cdot\mathbf{G}=\sum_{n_1,\varpi_1}\sum_{n_2,\varpi_2}\frac{1}{2}\left[\widehat{\mathbf{G}}_{n_1}^{\sim\varpi_1}\varphi_{n_1}^{j_2}(t)\right]\cdot\left[\widehat{\mathbf{G}}_{n_2}^{\sim\varpi_2}\varphi_{n_2}^{j_2}(t)\right]. \tag{6.5}$$

So the flux of the "atom" over $t\in[0,1]$ at step n on window ϖ is

$$\int_0^1\mathbf{v}(t)\ \frac{1}{2}\left[\widehat{\mathbf{G}}_{n_1}^{\sim\varpi_1}\varphi_{n_1}^{j_2}(t)\right]\cdot\left[\widehat{\mathbf{G}}_{n_2}^{\sim\varpi_2}\varphi_{n_2}^{j_2}(t)\right]\ \delta(n-n_2)\delta(\varpi-\varpi_2)dt, \tag{6.6}$$

and the flux of K by $\mathbf{v}$ on ϖ at n is

$$\mathbf{Q}_n^{\varpi}=\sum_{n_1,\varpi_1}\sum_{n_2,\varpi_2}\frac{1}{2}\mathbf{v}(t)\left[\widehat{\mathbf{G}}_{n_1}^{\sim\varpi_1}\varphi_{n_1}^{j_2}(t)\right]\cdot\left[\widehat{\mathbf{G}}_{n_2}^{\sim\varpi_2}\varphi_{n_2}^{j_2}(t)\right]\ \delta(n-n_2)\delta(\varpi-\varpi_2)dt$$

$$= \frac{1}{2}\int_0^1 [\mathbf{v}(t)\mathbf{G}(t)] \cdot \widehat{\mathbf{G}}_n^{\sim\varpi} \varphi_n^{j_2}(t)dt, \tag{6.7}$$

where the dyadic $\mathbf{vG}$ takes right dot product with $\widehat{\mathbf{G}}_n^{\sim\varpi}$. Again, $\widehat{\mathbf{G}}_n^{\sim\varpi}\varphi_n^{j_2}(t)$ lies in window ϖ. Due to the orthogonality among windows,

$$\begin{aligned}
\mathbf{Q}_n^{\varpi} &= \frac{1}{2}\int_0^1 [\mathbf{v}(t)\mathbf{G}(t)]^{\sim\varpi} \cdot \widehat{\mathbf{G}}_n^{\sim\varpi} \varphi_n^{j_2}(t)dt \\
&= \frac{1}{2}\widehat{(\mathbf{vG})}_n^{\sim\varpi} \cdot \widehat{\mathbf{G}}_n^{\sim\varpi} \\
&= \frac{1}{2}\left[\widehat{(\mathbf{v}G_1)}_n^{\sim\varpi}\widehat{(G_1)}_n^{\sim\varpi} + \widehat{(\mathbf{v}G_2)}_n^{\sim\varpi}\widehat{(G_2)}_n^{\sim\varpi}\right],
\end{aligned} \tag{6.8}$$

which is like the superposition of the fluxes of two scalar fields, namely, G_1 and G_2."

6.2. Canonical transfer with MWT. Given the multiscale flux, it is now easy to find the energy transfer. The following is from Liang (2016), section 3b.

"Consider a scalar property T in an incompressible flow field $\mathbf{v}$. The equation governing the evolution of T is

$$\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{v}T) = \text{other terms.}$$

As only the nonlinear term namely the advection will lead to interscale transfer, all other terms are unexpressed and put to the right hand side. To find its evolution on window ϖ, take MWT on both sides. The first term is $\widehat{\left(\frac{\partial T}{\partial t}\right)}_n^{\sim\varpi}$. It has been shown by LR05 to be approximately equal to $\frac{\delta \widehat{T}_n^{\sim\varpi}}{\delta n}$, where $\frac{\delta}{\delta n}$ is the difference operator with respect to n. Since t of the physical space is now carried over to n of the sampling space, the difference operator is essentially the time rate of change when applying to a discrete time series. We therefore would write it as $\frac{\partial \widehat{T}_n^{\sim\varpi}}{\partial t}$ to avoid introducing extra notations, which are already too many. But the careful reader should bear in mind that here it means the difference in the sampling space rather than the differential in the physical space. (Since the signals are sampled at each time step, in real applications they are precisely the same.) The MWTed equation is, therefore,

$$\frac{\partial \widehat{T}_n^{\sim\varpi}}{\partial t} + \nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} = ...$$

Multiplication of $\widehat{T}_n^{\sim\varpi}$ gives

$$\frac{\partial E_n^{\varpi}}{\partial t} = -\widehat{T}_n^{\sim\varpi}\nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} + ... \tag{6.9}$$

where $E_n^{\varpi} = \frac{1}{2}\left(\widehat{T}_n^{\sim\varpi}\right)^2$ is the energy on window ϖ at step n.

One continuing effort in multiscale energetics study is to separate $-\widehat{T}_n^{\sim\varpi}\nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi}$ into a transport process term and a transfer process term. Symbolically this is

$$-\nabla \cdot \mathbf{Q}_n^{\varpi} + \Gamma_n^{\varpi}.$$

An intuitively and empirically based common practice is to collect divergence terms to form the transport term (e.g., Harrison and Robinson, 1978; Pope, 2003). However, as long pointed by people such as Holopainen (1978), Plumb (1983), among others, there exist other forms that may result in different separations.

In this study, the separation is natural. The multiscale flux $\mathbf{Q}_n^{\varpi}$, hence the multiscale transport, has been rigorously obtained in the preceding subsection! The transfer Γ is obtained by subtracting $-\nabla \cdot \mathbf{Q}_n^{\varpi}$ from the right hand side of (6.9):

$$\begin{aligned} \Gamma_n^{\varpi} &= -\widehat{T}_n^{\sim\varpi} \nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} + \nabla \cdot \left[\frac{1}{2} \widehat{T}_n^{\sim\varpi} \widehat{(\mathbf{v}T)}_n^{\sim\varpi} \right] \\ &= \frac{1}{2} \left[\widehat{(\mathbf{v}T)}_n^{\sim\varpi} \cdot \nabla \widehat{T}_n^{\sim\varpi} - \widehat{T}_n^{\sim\varpi} \nabla \cdot \widehat{(\mathbf{v}T)}_n^{\sim\varpi} \right]. \end{aligned} \tag{6.10}$$

Notice that the resulting transfer bears a form similar to the Lie bracket and, particularly, the Poisson bracket in Hamiltonian mechanics. Because of this, we will refer it to as *canonical transfer* in the future, in order to distinguish it from other transfers already existing in the literature.

Canonical transfers possess a very important property, as stated in the following theorem.

THEOREM 6.1. *A canonical transfer vanishes upon summation over all the scale windows and marginalization over the sampling space, i.e.,*

$$\sum_n \sum_{\varpi} \Gamma_n^{\varpi} = 0. \tag{6.11}$$

Remark: This theorem states that a canonical transfer process only re-distributes energy among scale windows, without generating or destroying energy as a whole. This is precisely one would expect for an energy transfer process!

Proof. By the property of marginalization (4.15), Eq. (6.10) gives

$$\sum_n \Gamma_n^{\varpi} = \frac{1}{2} \int_0^1 \left[(\mathbf{v}T)^{\sim\varpi} \cdot \nabla T^{\sim\varpi} - T^{\sim\varpi} \nabla \cdot (\mathbf{v}T)^{\sim\varpi} \right] \, dt.$$

Because of the orthogonality between different scale windows, this followed by a summation over ϖ results in

$$\frac{1}{2} \int_0^1 \left[(\mathbf{v}T) \cdot \nabla T - T \nabla \cdot (\mathbf{v}T) \right] \, dt = 0.$$

In the above derivation, the incompressibility assumption of the flow has been used.

The canonical transfer (6.10) may be further simplified in expression when $\widehat{T}_n^{\sim\varpi}$ is nonzero:

$$\Gamma_n^{\varpi} = -E_n^{\varpi} \; \nabla \cdot \left(\frac{\widehat{(\mathbf{v}T)}_n^{\sim\varpi}}{\widehat{T}_n^{\sim\varpi}} \right), \text{ if } \widehat{T}_n^{\sim\varpi} \neq 0, \tag{6.12}$$

where $E_n^{\varpi} = \frac{1}{2} \left(\widehat{T}_n^{\sim\varpi} \right)^2$ is the energy on window ϖ at step n, and is hence always positive. Note that (6.12) defines a field variable which has the dimension of velocity in physical space:

$$\mathbf{v}_T^{\varpi} = \frac{\widehat{(T\mathbf{v})}_n^{\sim\varpi}}{\widehat{T}_n^{\sim\varpi}}. \tag{6.13}$$

It may be loosely understood as a weighted average of $\mathbf{v}$, with the weights derived from the MWT of the scalar field T. For convenience, we will refer to $\mathbf{v}_T^{\varpi}$ as T-*coupled velocity*. The growth rate of energy on window ϖ is now totally determined by $-\nabla \cdot \mathbf{v}_T^{\varpi}$, the convergence of $\mathbf{v}_T^{\varpi}$, and

$$\Gamma_n^{\varpi} = -E_n^{\varpi} \nabla \cdot \mathbf{v}_T^{\varpi}. \tag{6.14}$$

Note Γ_n^{ϖ} makes sense even when $\widehat{T}_n^{\sim\varpi} = 0$ and hence $\mathbf{v}_T^{\varpi}$ does not exist. In this case, (6.14) should be understood as (6.10)."

6.3. Canonical transfer with Reynolds decomposition. Previously we have shown that the Reynolds decomposition is a very special case of MWT when $j_0 = 0$ and periodization is used for extension scheme. One naturally wants to know how canonical transfer (6.14) or (6.10) may appear in that special case, as this will allow us to see how it differs from the classical energy transform in (5.3). As when $j_0 = 0$ and a periodization scheme is used, $u^{\sim 0} = \bar{u}$ $u^{\sim j_1} = u'$ (Liang and Anderson, 2007), it is easy to prove that (Liang and Robinson, 2007; Liang, 2016)

$$\Gamma_n^1 = \frac{1}{2}\left[\bar{T}\nabla \cdot (\overline{\mathbf{v}'T'}) - (\overline{\mathbf{v}'T'})\nabla\bar{T}\right] \equiv \Gamma, \tag{6.15}$$

$$\Gamma_n^0 = -\Gamma. \tag{6.16}$$

Accordingly the energetic equations become

$$\frac{\partial \bar{T}^2/2}{\partial t} + \nabla \cdot \left(\frac{1}{2}\bar{\mathbf{v}}\bar{T}^2 + \frac{1}{2}\bar{T}\overline{\mathbf{v}'T'}\right) = -\Gamma, \tag{6.17a}$$

$$\frac{\partial \overline{T'^2/2}}{\partial t} + \nabla \cdot \left(\frac{1}{2}\overline{\mathbf{v}T'^2/2} + \frac{1}{2}\bar{T}\overline{\mathbf{v}'T'}\right) = \Gamma. \tag{6.17b}$$

For easy convenience, we re-write here the classical energy equations, i.e., (5.3):

$$\frac{\partial \bar{T}^2/2}{\partial t} + \nabla \cdot (\bar{\mathbf{v}}\bar{T}^2/2) = -\bar{T}\nabla \cdot (\overline{\mathbf{v}'T'}),$$

$$\frac{\partial \overline{T'^2/2}}{\partial t} + \nabla \cdot (\overline{\mathbf{v}T'^2/2}) = -\overline{\mathbf{v}'T'} \cdot \nabla\bar{T},$$

and compare it to (6.17). The difference is clearly seen. In (6.17), the right hand sides are now balanced, and energy is conserved—This is how canonical transfer earns its name. Note we did not purportedly require that the right hand sides sum to zero; we derived the equations in a rigorous way and they naturally appear this way.

6.4. Validation with Kuo's barotropic instability model. In this section a benchmark instability model with analytical solution is used to validate the afore-obtained formula of canonical transfer. For clarity, we just consider the simplest case, i.e., the case with the most special case of MWT—the Reynolds decomposition. But before doing that, we need to extend (6.15), the formula with a passive scalar equation, to a setting with momentum equation. This is easy to fulfill. One just needs to take each component of $\mathbf{v}$ as the scalar field to compute the canonical transfer (6.15), then sum all the resulting transfers. The resulting total canonical transfer is

$$\Gamma = \frac{1}{2}\left\{\nabla \cdot (\overline{\mathbf{v}'\mathbf{v}'}) \cdot \bar{\mathbf{v}} - (\overline{\mathbf{v}'\mathbf{v}'}) : \nabla\bar{\mathbf{v}}\right\}, \tag{6.18}$$

where the colon operator "**:**" is defined such that, for two dyadic products **AB** and **CD**, $(\mathbf{AB}) : (\mathbf{CD}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D})$. The detailed derivation is referred to Liang (2016).

Note the second part in (6.18),

$$-(\overline{\mathbf{v}'\mathbf{v}'}) : \nabla\bar{\mathbf{v}} \equiv \mathcal{R},$$

is the classical Reynolds stress extraction of energy against the basic profile, which has been used widely to interpret mean-eddy interaction, turbulence production, etc. We will see through this concrete example how Γ differs from it.

Consider a well-studied barotropic instability model, the Kuo model, for the instability of the zonal atmospheric jet stream (Kuo, 1948). Liang and Robinson (2007) have constructed a particular solution with a highly localized structure which is ideal for our purpose here. In the following we briefly present this solution, and then calculate the transfer (6.18).

Choose a coordinate frame with x pointing eastward, y northward. The domain is periodic in x, and limited within latitudes $y = \pm L$, where a slip boundary condition $v = 0$ is applied. Assume a basic velocity profile (cf. Fig. 6.1a)

$$\bar{u}(y) = \bar{u}_{\max} \cos^2\left(\frac{\pi}{2}\frac{y}{L}\right), \qquad \bar{u}_{\max} > 0. \tag{6.19}$$

The background potential vorticity q has a meridional gradient (cf. Fig. 6.1b)

$$\bar{q}_y = -\bar{u}_{yy} = -\frac{\pi^2}{2L^2}\bar{u}_{\max} \cos\frac{\pi y}{L}, \tag{6.20}$$

which changes sign at $y = \pm\frac{L}{2}$, meeting the necessary condition for instability by Rayleigh's theorem (*ibid*). Decompose the flow as

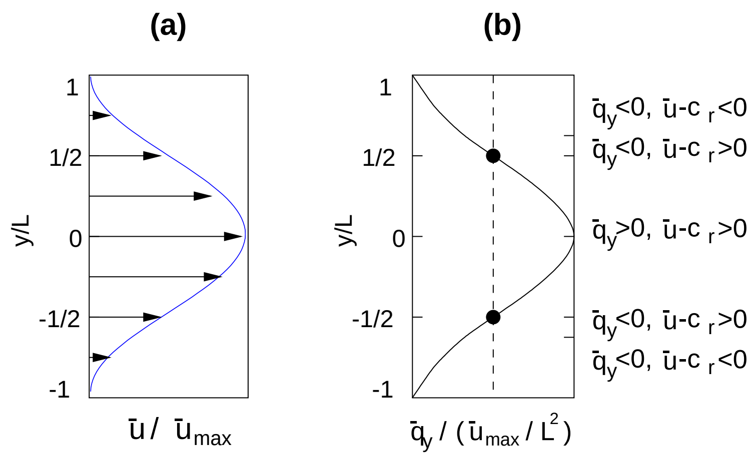

Fig. 6.1. *Configuration of the Kuo model. (a) The basic flow profile $\bar{u} = \bar{u}(y)$. (b) The background potential vorticity. Marked are the two reflection points on the profile curve.*

$$(u, v) = (\bar{u}(y), 0) + (u', v'), \tag{6.21}$$

and substitute back to the governing equations. Kuo considered only the initial stage of instability when the perturbation field (u', v') is very small. So the resulting equations can be linearized. Assuming a solution of the form

$$(u', v') = (\tilde{u}(y), \tilde{v}(y))e^{ik(x-ct)}, \tag{6.22}$$

Kuo (1948) obtained an eigenvalue problem

$$\frac{d^2\tilde{v}}{dy^2} + \left(\frac{\bar{u}_y y}{c - \bar{u}} - k^2\right)\tilde{v} = 0, \tag{6.23}$$

with boundary conditions

$$\tilde{v} = 0, \qquad \text{at } y = \pm L. \tag{6.24}$$

The solution of (6.23) is not repeated here; the reader may refer to Kuo's original papers for details.

Kuo showed that, in addition to the q_y inflection requirement, the difference $(\bar{u} - c_r)$ ($c_r = \text{Re}\{c\}$ the mode phase velocity) must be positively correlated with $\bar{q}_y$ over $[-L, L]$ in order for the perturbation to destabilize the jet. In other words, for an instability to occur, it requires that

(1) $\bar{q}_y$ change sign through $y \in [-L, L]$ (Rayleigh' theorem);

(2) $(\bar{u} - c_r)$ and $\bar{q}_y$ be positively correlated over $[-L, L]$ (Kuo's theorem).

Hence the zero points of $(\bar{u} - c_r)$ and $\bar{q}_y$ are critical. We will validate our transfer formalism through examining the instability structures near these critical points. We choose a particular unstable mode (and hence a particular c_r) to fulfill the objective.

As shown in Liang and Robinson (2007), the wavenumber $k = \frac{3}{4}\frac{\pi}{L}$ gives such a mode; it lies within the unstable regime as computed by Kuo (1948). In fact, if substituting back into the eigenvalue problem, one obtains, using the shooting method,

$$c = c_r + ic_i = (0.4504 + 0.0476i)\bar{u}_{\max}, \tag{6.25}$$

yielding a positive growth rate $kc_i > 0$. Solved in the mean time is the corresponding eigenvector $\tilde{v}$, which substituted in (6.22) and the governing equations give a solution of all the fields. The resulting phase speed $c_r = 0.4504\bar{u}_{\max}$ and the gradient of the basic potential vorticity $\bar{q}_y$ give four critical values of y:

$$\begin{aligned} &\bar{u}(y) - c_r = 0 &&\Longrightarrow y = \pm 0.53L, \\ &\bar{q}_y = -\bar{u}_{yy} = 0 &&\Longrightarrow y = \pm 0.50L. \end{aligned} \tag{6.26}$$

The four critical latitudes, as marked in Fig. 6.1b, partitions the y dimension into five distinct regimes characterized by different values of $K \equiv \bar{q}_y(\bar{u} - c_r)$. For most of $y \in [-L, l]$, $K > 0$, but the positivity is interrupted by two narrow strips near $y = \pm L/2$, where $K < 0$. This scenario has profound implications by Kuo's theorem. Although Kuo's theorem is stated in a global form, it should hold locally within the correlation scale. In the present example, that means one of the necessary conditions for barotropic instability is not met around the strips and so there should be no instability occurring there.

Instability means a transfer of energy from the background to the perturbation field, namely, a positive Γ. Using the particular solution obtained above, we compute the transfer from (6.18). We adopt a zonal averaging, i.e., averaging in x, to fulfill the decomposition. This is because, (1) $\bar{u}$ itself does not have x-dependence and hence can be understood as an x-average, and (2) the solution is homogeneous in x due to the cyclic boundary condition. The computation is straightforward. The result is plotted in Fig. 6.2a. Sure enough, Γ is not positive around the two narrow strips; in fact, there is a strong negative transfer, i.e., upscale or inverse transfer from the eddy window to the background. Moreover, the negative transfer is limited

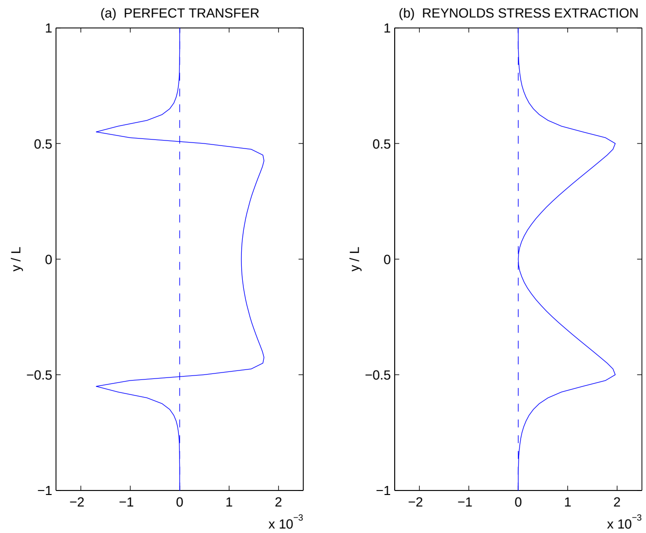

FIG. 6.2. *The barotropic energy transfer (scaled by $\bar{u}^3_{\max}/L$) for the Kuo's model: (a) the perfect transfer Γ, which is $\frac{1}{2}\left[\bar{u}\frac{\partial\overline{u'v'}}{\partial y} - \overline{u'v'}\frac{\partial\bar{u}}{\partial y}\right]$ here by (6.18); (b) the Reynolds stress extraction $\mathcal{R}$, which is equal to $-\overline{u'v'}\frac{\partial\bar{u}}{\partial y}$ here. The averaging is taken with respect to x.*

within two narrow regimes, just as one may expect by Kuo's theorem. In contrast, a different scenario is seen on the profile of the conventional Reynolds stress extraction $\mathcal{R}$, which we plot in Fig. 6.2b. $\mathcal{R}$ is nonnegative throughout $[-L, L]$; particularly, it is maximally positive over the narrow strip regimes, countering our foregoing intuitive argument. Through this example, our perfect transfer Γ results in a scenario agreeing well with the analytical result of the Kuo model, while the conventional Reynolds stress extraction $\mathcal{R}$ does not.

7. Multiscale energetics and localized Lorenz cycle. The above theory can be easily applied to derive the multiscale oceanic and atmospheric energetics. The following is just a brief summary of the results. Details are referred to Liang (2016). For notational brevity, from now on the *dependence on n will be suppressed* in the MWT terms, unless otherwise indicated.

7.1. Multiscale oceanic energetics. For an incompressible and hydrostatic Boussinesq fluid flow, the primitive equations are:

$$\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h + w\frac{\partial \mathbf{v}_h}{\partial z} + f\mathbf{k}\times\mathbf{v}_h = -\frac{1}{\rho_0}\nabla_h P + \mathbf{F}_{m,z} + \mathbf{F}_{m,h} \tag{7.1}$$

$$\frac{\partial P}{\partial z} = -\rho g \tag{7.2}$$

$$\nabla_h \cdot \mathbf{v}_h + \frac{\partial w}{\partial z} = 0 \tag{7.3}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{v}_h \cdot \nabla_h \rho + w\frac{\partial \rho}{\partial z} = \frac{N^2\rho_0}{g} w + F_{\rho,z} + F_{\rho,h}, \tag{7.4}$$

where the subgrid process parameterization are symbolically written as $\mathbf{F}_m$ and F_ρ. The symbols are conventional. The MWT-based multiscale ocean energetic equations were first derived by Liang and Robinson (2005), and updated in Liang (2016). They can be written as

$$\frac{\partial A^\varpi}{\partial t} + \nabla \cdot Q_A^\varpi = \Gamma_A^\varpi + b^\varpi + S_A^\varpi + F_{A,z}^\varpi + F_{A,h}^\varpi, \tag{7.5}$$

$$\frac{\partial K^\varpi}{\partial t} + \nabla \cdot Q_K^\varpi = \Gamma_{K^\varpi} - \nabla \cdot \mathbf{Q}_P^\varpi - b^\varpi + F_{K,z}^\varpi + F_{K,h}^\varpi, \tag{7.6}$$

with the symbol expressions tabulated in Table 7.1. It should be mentioned that all the terms are to be multiplied by a constant factor 2^{j_2}, where j_2 is the upper bound of the scale level of the smallest scale window. Refer to Liang (2016) for expressions in spherical coordinates.

Table 7.1

Multiscale energetics for oceanic circulations (Liang 2016). The units are in $m^2 s^{-3}$ (if SI units are used). The colon operator ":" is defined such that, for two dyadic products $\mathbf{AB}$ *and* $\mathbf{CD}$, $(\mathbf{AB}) : (\mathbf{CD}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D})$. *All terms are to be multiplied by 2^{j_2}, with j_2 the highest scale level of the series in question.*

K^ϖ	$\frac{1}{2}\widehat{\mathbf{v}}_h^{\sim\varpi} \cdot \widehat{\mathbf{v}}_h^{\sim\varpi}$	KE on scale window ϖ
$\mathbf{Q}_K^\varpi$	$\frac{1}{2}[\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} \cdot \widehat{\mathbf{v}}_h^{\sim\varpi}]$	flux of KE on window ϖ
Γ_K^ϖ	$\frac{1}{2}[\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} : \nabla\widehat{\mathbf{v}}_h^{\sim\varpi} - \nabla \cdot \widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} \cdot \widehat{\mathbf{v}}_h^{\sim\varpi}]$	canonical transfer of KE to window ϖ
$\mathbf{Q}_P^\varpi$	$\frac{1}{\rho_0}\widehat{\mathbf{v}}^{\sim\varpi}\widehat{P}^{\sim\varpi}$	pressure flux on window ϖ
b^ϖ	$\frac{g}{\rho_0}\widehat{\rho}^{\sim\varpi}\widehat{w}^{\sim\varpi}$	buoyancy conversion on window ϖ
A^ϖ	$\frac{1}{2}c(\widehat{\rho}^{\sim\varpi})^2, \quad c = \frac{g^2}{\rho_0^2 N^2}$	APE on window ϖ
$\mathbf{Q}_A^\varpi$	$\frac{1}{2}[c\widehat{\rho}^{\sim\varpi}\widehat{(\mathbf{v}\rho)}^{\sim\varpi}]$	flux of APE on window ϖ
Γ_A^ϖ	$\frac{c}{2}[\widehat{(\mathbf{v}\rho)}^{\sim\varpi} \cdot \nabla\widehat{\rho}^{\sim\varpi} - \widehat{\rho}^{\sim\varpi}\nabla \cdot \widehat{(\mathbf{v}\rho)}^{\sim\varpi}]$	canonical transfer of APE to window ϖ
S_A^ϖ	$\frac{1}{2}\widehat{\rho}^{\sim\varpi}\widehat{(\omega\rho)}^{\sim\varpi}\frac{\partial c}{\partial z}$	apparent source/sink of A^ϖ (usually negligible)

7.2. Multiscale atmospheric energetics. Consider an ideal gas and assume hydrostaticity to hold. In isobaric coordinates, the governing equations are:

$$\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h + \omega \frac{\partial \mathbf{v}_h}{\partial p} + f\mathbf{k} \times \mathbf{v_h} = -\nabla_h \Phi^* + \mathbf{F}_{m,p} + \mathbf{F}_{m,h}, \tag{7.7}$$

$$\frac{\partial \Phi^*}{\partial p} = -\alpha^*, \tag{7.8}$$

$$\nabla_h \cdot \mathbf{v}_h + \frac{\partial \omega}{\partial p} = 0, \tag{7.9}$$

$$\frac{\partial T^*}{\partial t} + \mathbf{v}_h \cdot \nabla_h T^* + \omega \frac{\partial T^*}{\partial p} - \frac{\alpha^* \omega}{c_p} = \frac{\dot{q}_{net}}{c_p}, \tag{7.10}$$

$$p\alpha^* = RT^*, \tag{7.11}$$

where $\dot{q}_{net}$ stands for the heating rate from all diabatic sources, $\omega = \frac{dp}{dt}$, and the starred variables mean the whole fields, with the corresponding non-starred ones reserved for their anomalies. The subscript h indicates the horizontal component on the p plane; for example, $\mathbf{v} = (\mathbf{v}_h, \omega)$, $\nabla = (\nabla_h, \frac{\partial}{\partial p})$, and so forth. The other symbols are conventional. With them Liang (2016) obtained the multiscale kinetic and available energy equations as follows:

$$\frac{\partial K^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_K^\varpi = \Gamma_K^\varpi - \nabla\cdot\mathbf{Q}_P^\varpi - b^\varpi + F_{K,p}^\varpi + F_{K,h}^\varpi, \tag{7.12}$$

$$\frac{\partial A^\varpi}{\partial t} + \nabla\cdot\mathbf{Q}_A^\varpi = \Gamma_A^\varpi + b^\varpi + S_A^\varpi + F_A^\varpi. \tag{7.13}$$

where the expressions for the energetics are listed in Table 7.2. Again, all the terms are to be multiplied by a constant factor 2^{j_2}, with j_2 being the upper bound of the scale level of the smallest scale window. Also tabulated are the expressions in spherical coordinates (λ, φ, p) (Table 7.3).

Table 7.2

Multiscale energetics for the atmospheric circulation (Liang 2016). The units are in $m^2 s^{-3}$ if SI base units are used. If total or regional total energies (in W) are to be computed, the resulting integrals with respect to (x, y, p) should be divided by g. All terms are to be multiplied by 2^{j_2}.

K^ϖ	$\frac{1}{2}\widehat{\mathbf{v}}_h^{\sim\varpi}\cdot\widehat{\mathbf{v}}_h^{\sim\varpi}$	KE on scale window ϖ
$\mathbf{Q}_K^\varpi$	$\frac{1}{2}\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}\cdot\widehat{\mathbf{v}}_h^{\sim\varpi}$	flux of KE on window ϖ
Γ_K^ϖ	$\frac{1}{2}[\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi} : \nabla\widehat{\mathbf{v}}_h^{\sim\varpi} - \nabla\cdot\widehat{(\mathbf{v}\mathbf{v}_h)}^{\sim\varpi}\cdot\widehat{\mathbf{v}}_h^{\sim\varpi}]$	canonical transfer of KE to window ϖ
$\mathbf{Q}_P^\varpi$	$\widehat{\mathbf{v}}^{\sim\varpi}\widehat{\Phi}^{\sim\varpi}$	pressure flux
b^ϖ	$\widehat{\omega}^{\sim\varpi}\widehat{\alpha}^{\sim\varpi}$	buoyancy conversion
A^ϖ	$\frac{1}{2}c(\widehat{T}^{\sim\varpi})^2, \quad c = \frac{g}{\bar{T}(g/c_p - L)}$	APE on scale window ϖ
$\mathbf{Q}_A^\varpi$	$\frac{1}{2}c\widehat{T}^{\sim\varpi}\widehat{(\mathbf{v}T)}^{\sim\varpi}$	flux of APE on window ϖ
Γ_A^ϖ	$\frac{c}{2}[\widehat{(\mathbf{v}T)}^{\sim\varpi}\cdot\nabla\widehat{T}^{\sim\varpi} - \widehat{T}^{\sim\varpi}\nabla\cdot\widehat{(\mathbf{v}T)}^{\sim\varpi}]$	canonical transfer of APE to window ϖ
S_A^ϖ	$\frac{1}{2}\widehat{T}^{\sim\varpi}\widehat{(\omega T)}^{\sim\varpi}\frac{\partial c}{\partial p} + \frac{1}{\bar{T}}\widehat{(\omega\alpha)}^{\sim\varpi}$	apparent source/sink (usually negligible)

Table 7.3

Expansion of the canonical transfers in Table 7.2 in spherical coordinates (Liang 2016).

Γ_K^ϖ	$\frac{1}{2a\cos\varphi}\left[\widehat{(u^2)}^{\sim\varpi}\frac{\partial\widehat{u}^{\sim\varpi}}{\partial\lambda} - \widehat{u}^{\sim\varpi}\frac{\partial\widehat{(u^2)}^{\sim\varpi}}{\partial\lambda} + \widehat{(uv)}^{\sim\varpi}\frac{\partial\widehat{v}^{\sim\varpi}}{\partial\lambda} - \widehat{v}^{\sim\varpi}\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\lambda}\right]$
	$+\frac{1}{2a}\left[\widehat{(uv)}^{\sim\varpi}\frac{\partial\widehat{u}^{\sim\varpi}}{\partial\varphi} - \widehat{u}^{\sim\varpi}\frac{\partial\widehat{(uv)}^{\sim\varpi}}{\partial\varphi} + \widehat{(v^2)}^{\sim\varpi}\frac{\partial\widehat{v}^{\sim\varpi}}{\partial\varphi} - \widehat{v}^{\sim\varpi}\frac{\partial\widehat{(v^2)}^{\sim\varpi}}{\partial\varphi}\right]$
	$+\frac{1}{2}\left[\widehat{(u\omega)}^{\sim\varpi}\frac{\partial\widehat{u}^{\sim\varpi}}{\partial p} - \widehat{u}^{\sim\varpi}\frac{\partial\widehat{(u\omega)}^{\sim\varpi}}{\partial p}\widehat{(v\omega)}^{\sim\varpi}\frac{\partial\widehat{v}^{\sim\varpi}}{\partial p} - \widehat{v}^{\sim\varpi}\frac{\partial\widehat{(v\omega)}^{\sim\varpi}}{\partial p}\right]$
	$+\frac{3}{2a}\tan\varphi\left[\widehat{u}^{\sim\varpi}\widehat{(uv)}^{\sim\varpi} - \widehat{v}^{\sim\varpi}\widehat{(u^2)}^{\sim\varpi}\right]$
	$+\frac{1}{2a}\widehat{v}^{\sim\varpi}\tan\varphi\left[\widehat{(u^2)}^{\sim\varpi} + \widehat{(v^2)}^{\sim\varpi}\right]$
Γ_A^ϖ	$\frac{c}{2}\left[\frac{1}{a\cos\varphi}\widehat{(uT)}^{\sim\varpi}\frac{\partial\widehat{T}^{\sim\varpi}}{\partial\lambda} + \frac{1}{a}\widehat{(vT)}^{\sim\varpi}\frac{\partial\widehat{T}^{\sim\varpi}}{\partial\varphi} + \widehat{(\omega T)}^{\sim\varpi}\frac{\partial\widehat{T}^{\sim\varpi}}{\partial p}\right.$
	$\left.-\frac{1}{a\cos\varphi}\widehat{T}^{\sim\varpi}\frac{\partial\widehat{(uT)}^{\sim\varpi}}{\partial\lambda} - \frac{1}{a\cos\varphi}\widehat{T}^{\sim\varpi}\frac{\partial[\widehat{(vT)}^{\sim\varpi}\cos\varphi]}{\partial\varphi} - \widehat{T}^{\sim\varpi}\frac{\partial\widehat{(\omega T)}^{\sim\varpi}}{\partial p}\right]$

7.3. Localized Lorenz cycles. Based on the above results, the energy cycle, or Lorenz cycle as called, can be localized, since each energetic term is essentially a 4D field variable due to the localized nature of MWT. A localized Lorenz cycle for an ideal fluid without external forcing is made of the following four types of conservative processes:

- transports which integrate to zero over closed domain;

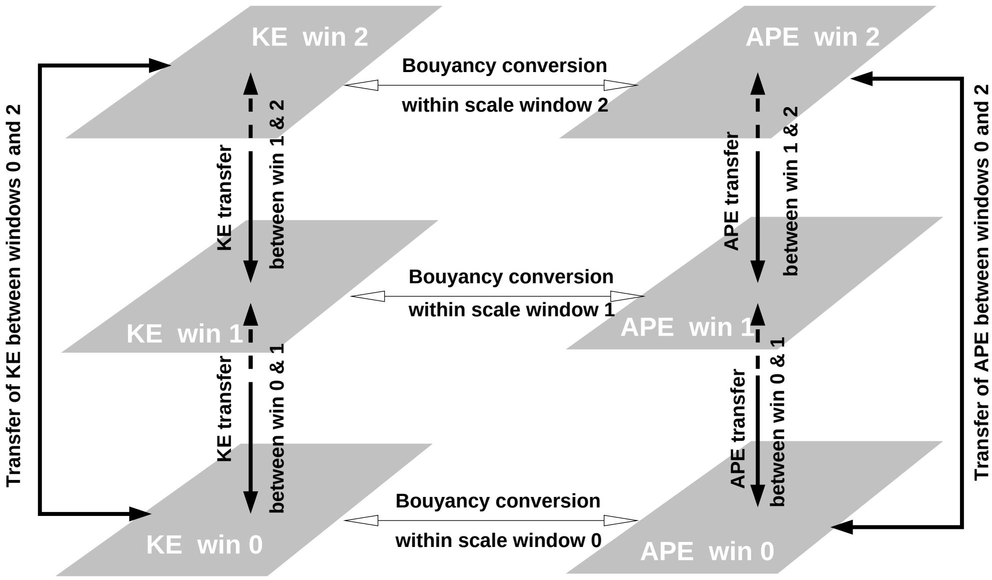

Fig. 7.1. *Schematic of energy transfer and buoyancy conversion for (a) two-window and (b) three-window decomposition. Clearly buoyancy conversions are not indicators of instabilities (from Liang 2016).*

- canonical transfers which sum to zero over all scale windows and sampling space locations;
- buoyancy conversions which sum to zero over mechanical energy type (e.g., KE or APE).

The partial cycle for a three-scale window decomposition is schematized in Fig. 7.1. For clarity, transport and some other processes are not drawn here. As is seen, canonical transfers mediate between the scale windows; they represent the interscale processes such as instabilities. In contrast, buoyancy conversions and transports function only within the respective individual windows; the former bring together the two types of energy, namely, APE and KE, while that latter allow different spatial locations to communicate.

PART III. EXEMPLARY APPLICATIONS

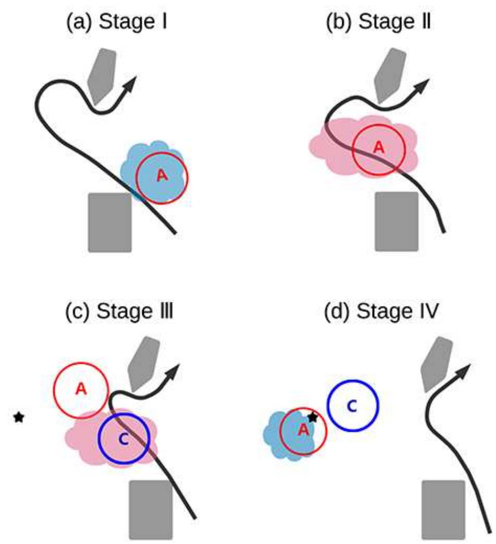

FIG. 8.1. *An anticyclonic eddy shed into the South China Sea through the Luzon Strait (Zhao et al., 2016).*

8. Eddy shedding into the South China Sea through Luzon Strait.

9. Kuroshio dynamics in East China Sea.

10. Kuroshio Extension.

11. Sudden stratospheric warming in the 2012/13 winter.

12. Atmospheric blocking over the Atlantic Ocean.

13. On the inverse relationship between the jet stream strength and storm intensity in the Pacific storm track.